Scale-invariant solutions to partial differential equations of fractional order with a moving boundary condition

2008 J. Phys. A: Math. Theor. 41 155202

(http://iopscience.iop.org/1751-8121/41/15/155202)

J. Phys. A: Math. Theor. **41** (2008) 155202 (11pp) doi:10.1088/1751-8113/41/15/155202

Scale-invariant solutions to partial differential equations of fractional order with a moving boundary condition

Xicheng Li[1], **Mingyu Xu**[1] **and Shaowei Wang**[2]

[1] Institute of Applied Mathematics, School of Mathematics and System Science, Shandong University, Jinan 250100, People's Republic of China
[2] Department of Mechanics and Engineering Science, Peking University, Beijing 100871, People's Republic of China

E-mail: xichengli@yahoo.com.cn

Received 18 October 2007, in final form 23 November 2007
Published 2 April 2008
Online at stacks.iop.org/JPhysA/41/155202

Abstract
In this paper, we give similarity solutions of partial differential equations of fractional order with a moving boundary condition. The solutions are given in terms of a generalized Wright function. The time-fractional Caputo derivative and two types of space-fractional derivatives are considered. The scale-invariant variable and the form of the solution of the moving boundary are obtained by the Lie group analysis. A comparison between the solutions corresponding to two types of fractional derivative is also given.

PACS numbers: 02.30.Jr, 02.60.Lj, 02.50.−r

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Fractional calculus [1, 2] is very important in describing the evolution of complex systems for the reason that most of the processes associated with complex systems have dynamics involving long memory in time and non-local in space and the fractional derivatives do have some of those characteristics. For example, the Caputo, the Riemann–Liouville and the Reisz fractional derivatives are widely used in the description of viscoelastic materials [3, 4, 5] and anomalous diffusions [6–8]. A survey paper on the application of fractional derivatives in modern mechanics is given by Xu and Tan [9].

In this paper, we will consider the application of the fractional diffusion equation (FDE) in the release of a solute from a polymer matrix in which the initial loading is higher than the solubility. It is a typical moving boundary problem from the point of view of mathematics. Though different types of anomalous diffusion equation with appropriate initial and boundary conditions have been considered [8, 10, 11], due to the high nonlinearity of moving boundary problems and the fact that many of the useful properties of an ordinary derivative are not known to carry over analogously for the case of a fractional derivative operator, fractional calculus has scarcely been applied to such problems. Liu and Xu [13] first introduced a time-fractional diffusion equation with a moving boundary condition to a drug release process. They used the time FDE as the master equation and gave an exact solution. Li and Xu *et al* [14] used the spacetime FDE to describe the process of a solute release from a polymer matrix in which the initial solute loading is higher than the solubility and gave the exact solution in terms of the Fox-H function.

In order to describe the process of a solute release from a polymer matrix, we use the models analogous to those discussed in [12–14]. In this paper, the fractional diffusion equation

$$ {}_0^C D_T^{\alpha} c(X, T) = \mathscr{D}_0 D_X^{\beta} c(X, T) \tag{1} $$

is used to describe the mass transport of the dissolved solute in the polymer matrix, where ${}_0^C D_T^{\alpha}$ is the Caputo derivative, ${}_0 D_X^{\beta}$ is the space derivative and $\mathscr{D}$ is the diffusive coefficient. In [12, 13], the space derivative used is the classical second-order partial derivative. In [14], the space-fractional derivative is generalized to the Riesz derivative. However, from the point of view of Lé vy flights, the Riesz derivative would be inaccurate to model the diffusive process in a finite domain [7, 15]. To skip this problem, Benson *et al* [16], Liu *et al* [17], del-Castillo-Negrete [18] used the Riemann–Liouville derivative as the space-fractional derivative operator in researching the solute transport. Zhang *et al* [19] obtained the result that the Riemann–Liouville operator is problematic and could result in unphysical results for the solute transport in bounded domains. They used the Caputo derivative as the space-fractional derivative. In this paper, we will consider the space-fractional operator both in the Riemann–Liouville definition and Caputo one. In the following, we will use ${}_0^C D_X^{\beta}$ and ${}_0^{RL} D_X^{\beta}$ to denote the Caputo derivative and the Riemann–Liouville derivative, respectively. The boundary and initial conditions are given by

$$ c(0, T) = 0, \tag{2} $$

$$ c(s(T), T) = C_s, \tag{3} $$

$$ (C_0 - C_s) {}_0^C D_T^{\alpha} s(T) = \mathscr{D}_0 D_X^{\beta-1} c(X, T)|_{X=s(T)}, \tag{4} $$

$$ s(0) = 0, \tag{5} $$

where C_0 and C_s are the initial concentration of the solute distributed in the matrix and the solubility of the solute in the solvent, respectively, (2) is the perfect sink condition, (3) is the concentration of the solute at the diffusion front, $s(T)$ is the position of the diffusion front which moves inward as time progresses, (4) is the generalized mass conservation condition and (5) is the initial condition of $s(T)$.

By denoting R the scale of the polymer matrix and using reduced dimensionless variables defined as

$$ x = \frac{X}{R}, \qquad t = \left(\frac{\mathscr{D}}{R^{\beta}}\right)^{\frac{1}{\alpha}} T, \qquad C = \frac{c}{C_s}, \qquad S(t) = \frac{s}{R}, $$

we can get the governing equation and the boundary and initial conditions in non-dimensional form:

$$ {}_0^C D_t^{\alpha} C(x, t) = {}_0 D_x^{\beta} C(x, t), \qquad (0 \leqslant x \leqslant S(t), 0 < \alpha \leqslant 1 < \beta \leqslant 2) \tag{6} $$

$$ C(0, t) = 0, \tag{7} $$

$$C(S(t), t) = 1, \tag{8}$$

$$\eta_0^C D_t^\alpha S(t) = {}_0D_x^{\beta-1} C(x, t)|_{x=S(t)}, \tag{9}$$

$$S(0) = 0, \tag{10}$$

where $\eta = \frac{C_0 - C_s}{C_5}$ is a constant larger than 0.

According to the results of previous works [20], we can see that very few solutions to moving boundary problems in a closed form can be obtained. They are mainly taking the form of the single variable and are known as the similarity solution or scale-invariant solution. In this paper, by the aid of the Lie group method [21–23], we determined the similarity variable z, and the equations satisfied by the similarity solutions. Two sets of solutions are presented and the proof is given.

2. The equation for the scale-invariant solution

Using the definitions concerning the Lie group method listed in Luchko and Gorenflo [21], we have the following theorem.

Theorem 1. *The similarity transformations under which equations (6)–(10) are invariant are given by the expressions*

$$z = xt^{-\alpha/\beta}, \qquad C(x, t) = f(z) \quad \textit{and} \quad S(t) = pt^{\alpha/\beta}, \tag{11}$$

where p is a constant to be determined.

Proof. By Setting

$$\bar{x} = \lambda^a x, \qquad \bar{t} = \lambda^b t, \qquad \bar{C} = \lambda^c C, \qquad \bar{S} = \lambda^d S, \tag{12}$$

and after some simple calculations, the following results can be obtained:

$${}_0^C D_t^\alpha C(x, t) = \lambda^{b\alpha - c} {}_0^C D_{\bar{t}}^\alpha \bar{C}(\bar{x}, \bar{t}). \tag{13}$$

$${}_0^{RL} D_x^\beta C(x, t) = \lambda^{-c+a\beta} {}_0^{RL} D_{\bar{x}}^\beta \bar{C}(\bar{x}, \bar{t}). \tag{14}$$

$${}_0^C D_x^\beta C(x, t) = \lambda^{-c+a\beta} {}_0^C D_{\bar{x}}^\beta \bar{C}(\bar{x}, \bar{t}). \tag{15}$$

Consequently, the boundary conditions (7)–(10) are changed to $\lambda^c \bar{C}(0, \lambda^{-b}\bar{t}) = 0$, $\lambda^c \bar{C}(\lambda^{-d}\bar{s}, \lambda^{-b}\bar{t}) = 1$, and $\eta\lambda^{-d+b\alpha} {}_0^C D_{\bar{t}}^\alpha \bar{S} = \lambda^{-c+(\beta-1)a} {}_0D_{\bar{x}}^{\beta-1} \bar{C}(\bar{s}, \bar{t})|_{\lambda^{-a}\bar{x}=\lambda^{-d}\bar{S}}$, respectively. These relations imply that, in order to keep the invariants of scaling transformations, these following relations must hold,

$$b\alpha - c = a\beta - c, \qquad c = 0, \qquad c + (\beta - 1)a = -d + b\alpha, \qquad a = d. \tag{16}$$

From relation (16), we can obtain $\frac{a}{b} = \frac{\alpha}{\beta}$. Considering the Lie group method, the similarity variable z can be obtained. Noting that $\bar{C}(\bar{s}, \bar{t}) = 1$, i.e. $\lambda^c C(\lambda^d s, \lambda^b t) = 1$, letting $\lambda^b = t^{-1}$, we have $C(s(t)t^{-\frac{\alpha}{\beta}}, 1) = 1$. This implies that $s(t) \sim t^{\frac{\alpha}{\beta}}$, i.e. $s(t) = pt^{\frac{\alpha}{\beta}}$. □

Theorem 2. *The Caputo derivative* ${}_0^C D_t^\alpha$, $(0 < \alpha \leqslant 1)$ *of the function* $C(x, t) = f(z)$, $z = xt^{-\alpha/\beta}$ *is given by the relation*

$${}_0^C D_t^\alpha f(z) = t^{-\alpha} {}_* P_{\beta/\alpha}^{0,1-\alpha} f(z) = t^{-\alpha} K_{\beta/\alpha}^{0,1-\alpha} \left(-\frac{\alpha}{\beta} \frac{\mathrm{d}}{\mathrm{d}z} f(z) \right), \tag{17}$$

where ${}_*P^{0,1-\alpha}_{\beta/\alpha}$ *is the Caputo-type modification of the Erdélyi–Kober fractional differential operator and*

$$K^{\tau,\alpha}_{\delta}g(y) = \frac{\delta}{\Gamma(\alpha)}y^{\delta\tau}\int_y^{\infty}(u^{\delta}-y^{\delta})^{\alpha-1}u^{-\delta(\tau+\alpha-1)}g(u)\,\mathrm{d}u, \qquad \alpha > 0. \tag{18}$$

Proof. This theorem is just a particular case $\gamma = 0$ and a modification with respect to β of the results from [23]. However, for the readers and our convenience, we give a short proof. By using the new variable $u = x/s^{\alpha/\beta}$, we have

$$\begin{aligned}{}^C_0D^{\alpha}_t f(z) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial f\left(\frac{x}{s^{\alpha/\beta}}\right)}{\partial s}\,\mathrm{d}s\\ &= \frac{1}{\Gamma(1-\alpha)}\int_{\infty}^{z}\left(t-\left(\frac{x}{u}\right)^{\frac{\beta}{\alpha}}\right)^{-\alpha}\frac{\mathrm{d}f(u)}{\mathrm{d}u}\,\mathrm{d}u\\ &= \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\int_{\infty}^{z}(u^{\frac{\beta}{\alpha}}-z^{\frac{\beta}{\alpha}})^{-\alpha}u^{\beta}\frac{\mathrm{d}f(u)}{\mathrm{d}u}\,\mathrm{d}u.\end{aligned} \tag{19}$$

Equation (19) is exactly the expression of the Caputo-type modification of the Erdélyi–Kober fractional derivative operator defined by (17) and (18). □

Remark 1. In order to make the process of the proof easier in the following sections, we give a different form of relation (17) here,

$${}_*P^{0,1-\alpha}_{\beta/\alpha}f(z) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\int_0^1(1-s)^{-\alpha}\frac{\partial f\left(zs^{-\frac{\alpha}{\beta}}\right)}{\partial s}\,\mathrm{d}s. \tag{20}$$

This relation can be got using variable $s = \left(\frac{z}{u}\right)^{\frac{\beta}{\alpha}}$ in (19).

Using the above relations, we have:

Theorem 3. *The reduced form of equation (6) is given by*

$${}_*P^{0,1-\alpha}_{\beta/\alpha}f(z) = {}_0D^{\beta}_z f(z), \tag{21}$$

and the conditions become

$$f(0) = 0, \qquad f(p) = 1, \qquad \eta p\frac{\Gamma\left(1+\frac{\alpha}{\beta}\right)}{\Gamma\left(1+\frac{\alpha}{\beta}-\alpha\right)} = {}_0D^{\beta-1}_z f(z)|_{z=p}. \tag{22}$$

In theorem 3, ${}_0D^{\beta}_z$ and ${}_0D^{\beta-1}_z$ are the space-fractional operators which can be the Caputo or Riemann–Liouville derivatives distinguished by the marks C and RL.

3. Similarity solutions to the equations

The similarity solutions of equation (6) in some cases have been given, such as the time-fractional derivative in Riemann–Liouville sense [22], the time- and space-fractional derivatives both in Riemann–Liouville sense [21], the time-fractional derivative in Caputo sense [23]. In this paper, we considered two cases which have not been researched. They are: case 1, the time- and space-fractional derivatives both in Caputo sense; case 2, the time- and space-fractional derivatives in Caputo and Riemann–Liouville senses respectively.

Theorem 4. *As for the case 1, the similarity solution of equation (21) is*

$$f(z) = C_1 z W_{(-\alpha,1-\frac{\alpha}{\beta})(\beta,2)}(z^{\beta}), \tag{23}$$

where $W_{(\mu,a)(\nu,b)}(z)$ is the generalized Wright function defined by

$$W_{(\mu,a)(\nu,b)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(a+\mu k)\Gamma(b+\nu k)},\qquad \mu,\nu\in R,a,b\in C. \tag{24}$$

In the following analysis, the constraint $-1<\mu<0,\nu>0$ is used.

Proof. The convergence of solution (23) was given in [21]. Two important properties of the Caputo derivative ([2][chapter 2, property 2.16, p 95]) needed in the proof are

$$ {}_0^C D_z^{\nu} z^{\mu}=\frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\nu)}z^{\mu-\nu},\qquad \mu>1+[\nu], \tag{25}$$

and

$$ {}_0^C D_z^{\nu} z^{n}=0,\qquad n\in N,n<\nu. \tag{26}$$

We first apply the operator ${}_0^C D_z^{\beta}$ to the solution term by term and use the above two properties (25) and (26),

$$\begin{aligned}{}_0^C D_z^{\beta}\{C_1 z W_{(-\alpha,1-\frac{\alpha}{\beta})(\beta,2)}(z^{\beta})\}&={}_0^C D_z^{\beta}\left\{C_1\frac{z}{\Gamma\left(1-\frac{\alpha}{\beta}\right)}+C_1\sum_{k=1}^{\infty}\frac{z^{\beta k+1}}{\Gamma\left(1-\frac{\alpha}{\beta}-\alpha k\right)\Gamma(2+\beta k)}\right\}\\&=C_1\sum_{k=1}^{\infty}\frac{z^{\beta k+1-\beta}}{\Gamma\left(1-\frac{\alpha}{\beta}-\alpha k\right)\Gamma(2+\beta k-\beta)}\\&=C_1 z W_{(-\alpha,1-\frac{\alpha}{\beta}-\alpha)(\beta,2)}(z^{\beta}).\end{aligned}\tag{27}$$

Considering the definition of series representation of the generalized Wright function, $\frac{\partial f(zs^{-\frac{\alpha}{\beta}})}{\partial s}$ can be obtained in the series form,

$$\begin{aligned}\frac{\partial f\left(zs^{-\frac{\alpha}{\beta}}\right)}{\partial s}&=\frac{\partial}{\partial s}\left[C_1\sum_{k=0}^{\infty}\frac{z^{\beta k+1}s^{-\frac{\alpha}{\beta}(\beta k+1)}}{\Gamma\left(1-\frac{\alpha}{\beta}-\alpha k\right)\Gamma(2+\beta k)}\right]\\&=C_1 z s^{-\frac{\alpha}{\beta}-1}\sum_{k=0}^{\infty}\frac{z^{\beta k}s^{-\alpha k}}{\Gamma\left(-\frac{\alpha}{\beta}-\alpha k\right)\Gamma(2+\beta k)}.\end{aligned}\tag{28}$$

However, we cannot apply the integral $\int_0^1(1-s)^{-\alpha}\,\mathrm{d}s$ term by term to the power series (28) because of the divergence of the corresponding integrals. In Luchko and Gorenflo's paper [21], the integral representation of the generalized Wright function was used when the Erdélyi–Kober operator was applied to it and the Fox-H function was also used as an intermediate step to prove the convergence of the integral. In our proof, the representation of the Caputo-type modification of the Erdélyi–Kober operator (20) is considered and the Fox-H function is used just as did in [21]. According to the definition of the Fox-H function [24], series (36) can be written as

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{z^{\beta k}s^{-\alpha k}}{\Gamma\left(-\frac{\alpha}{\beta}-\alpha k\right)\Gamma(2+\beta k)}&=H_{2,2}^{1,1}\left[-z^{\beta}s^{-\alpha}\left|\begin{matrix}(0,1)(-\frac{\alpha}{\beta},\alpha)\\(0,1)(-1,\beta)\end{matrix}\right.\right]\\&=\frac{1}{2\pi\mathrm{i}}\int_L\frac{\Gamma(-\varsigma)\Gamma(1+\varsigma)}{\Gamma(2+\beta\varsigma)\Gamma\left(-\frac{\alpha}{\beta}-\alpha\varsigma\right)}(-z^{\beta}s^{-\alpha})^{\varsigma}\,\mathrm{d}\varsigma,\end{aligned}\tag{29}$$

where L is a contour separating the poles of $\Gamma(-\varsigma)$ and $\Gamma(1+\varsigma)$, i.e. $\{0,1,2,\ldots\}$ and $\{-1,-2,\ldots\}$. Here we choose L as a line from $a-\mathrm{i}\infty$ to $a+\mathrm{i}\infty$, provided that the condition

$-1 < a < -\frac{1}{\beta}$ holds. Then we have

$$
\begin{aligned}
{}_*P^{0,1-\alpha}_{\beta/\alpha} f(z) = {} & \frac{C_1 z}{\Gamma(1-\alpha)} \int_0^1 (1-s)^{-\alpha} s^{-\frac{\alpha}{\beta}-1} \\
& \cdot \frac{1}{2\pi \mathrm{i}} \int_L \frac{\Gamma(-\varsigma)\Gamma(1+\varsigma)}{\Gamma(2+\beta\varsigma)\Gamma\left(-\frac{\alpha}{\beta}-\alpha\varsigma\right)} (-z^\beta s^{-\alpha})^\varsigma \, \mathrm{d}\varsigma \, \mathrm{d}s.
\end{aligned} \tag{30}
$$

Noting that $\mathrm{Re}(-\alpha\varsigma - \frac{\alpha}{\beta}) > 0$ and the definition of the *beta* function

$$
B(a,b) = \int_0^1 (1-t)^{a-1} t^{b-1} \, \mathrm{d}t = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \qquad \mathrm{Re}(a), \mathrm{Re}(b) > 0, \tag{31}
$$

we can change the order of integration. As a result, (30) becomes,

$$
\begin{aligned}
& \frac{C_1 z}{2\pi \mathrm{i}} \int_L \frac{\Gamma(-\varsigma)\Gamma(1+\varsigma)}{\Gamma(2+\beta\varsigma)\Gamma\left(-\frac{\alpha}{\beta}-\alpha\varsigma\right)} (-z^\beta)^\varsigma \frac{1}{\Gamma(1-\alpha)} \int_0^1 (1-s)^{-\alpha} s^{-\alpha\varsigma-\frac{\alpha}{\beta}-1} \, \mathrm{d}s \, \mathrm{d}\varsigma \\
& \quad = \frac{C_1 z}{2\pi \mathrm{i}} \int_L \frac{\Gamma(-\varsigma)\Gamma(1+\varsigma)}{\Gamma(2+\beta\varsigma)\Gamma\left(1-\alpha-\frac{\alpha}{\beta}-\alpha\varsigma\right)} (-z^\beta)^\varsigma \, \mathrm{d}\varsigma \\
& \quad = C_1 z H^{1,1}_{2,2}\left[-z^\beta \left|^{(0,1)(1-\alpha-\frac{\alpha}{\beta},\alpha)}_{(0,1)(-1,\beta)}\right.\right].
\end{aligned} \tag{32}
$$

Taking the expansion for the Fox-H function [24],

$$
H^{m,n}_{p,q}(z) = \sum_{h=1}^{m} \sum_{k=0}^{\infty} \frac{\prod_{j=1,j\neq h}^{m} \Gamma\left(b_j - B_j \frac{b_h+k}{B_h}\right) \prod_{j=1}^{n} \Gamma\left(1-a_j + A_j \frac{b_h+k}{B_h}\right)}{\prod_{j=m+1}^{q} \Gamma\left(1-b_j + B_j \frac{b_h+k}{B_h}\right) \prod_{j=n+1}^{p} \Gamma\left(a_j - A_j \frac{b_h+k}{B_h}\right)} \frac{(-1)^k z^{(b_h+k)/B_h}}{k! B_h},
$$

(32) can be written as:

$$
\begin{aligned}
C_1 z H^{1,1}_{2,2}\left[-z^\beta \left|^{(0,1)(1-\alpha-\frac{\alpha}{\beta},\alpha)}_{(0,1)(-1,\beta)}\right.\right] &= C_1 z \sum_{k=0}^{\infty} \frac{\Gamma(1+k)}{\Gamma(2+\beta k)\Gamma(1-\alpha-\frac{\alpha}{\beta}-\alpha k)} \frac{(-1)^k(-z^\beta)^k}{k!} \\
&= C_1 z W_{(-\alpha,1-\frac{\alpha}{\beta}-\alpha)(\beta,2)}(z^\beta).
\end{aligned} \tag{33}
$$

Comparing (33) with (27), we can get the result. □

Remark 2. Both in this paper and the previous paper [21–23], detail descriptions of how to get the similarity solution of equation (21) were not given. Actually, there is no efficient method to solve this problem universally as yet. However, there are still two methods that may be considered. The first one is the Mellin transform method used by Wyss [25]. This method needs solving a differential equation which is a difficult job. The second one is to search the solution in series form $\sum_{k=0}^{\infty} a_k z^{b_k}$. By considering the boundary and initial conditions and property of the fractional derivatives (26), we can first consider the analytical continuation of the *beta* function for the entire complex plane [1] in order to get the value of a_k and b_k, and then verify the convergence and the validity of the series. In this paper, we choose the second method for simplicity.

By analogy with theorem 4, we have

Theorem 5. *As for the case 2, the similarity solution of equation (21) is given by*

$$
f(z) = C_2 z^{\beta-1} W_{(-\alpha,1+\frac{\alpha}{\beta}-\alpha)(\beta,\beta)}(z^\beta). \tag{34}
$$

Proof. Apply the operator ${}^{RL}_{0}D^{\beta}_{z}$ to solution (34) term by term, we have

$$
{}^{RL}_{0}D^{\beta}_{z}\{C_2 z^{\beta-1} W_{(-\alpha,1+\frac{\alpha}{\beta}-\alpha)(\beta,\beta)}(z^\beta)\} = C_2 z^{\beta-1} W_{(-\alpha,1+\frac{\alpha}{\beta}-2\alpha)(\beta,\beta)}(z^\beta). \tag{35}
$$

Considering the definition of series representation of the generalized Wright function, $\frac{\partial f\left(zs^{-\frac{\alpha}{\beta}}\right)}{\partial s}$ can be obtained in the series form,

$$\begin{aligned}
\frac{\partial f(zs^{-\frac{\alpha}{\beta}})}{\partial s} &= \frac{\partial}{\partial s}\left[C_2\sum_{k=0}^{\infty}\frac{z^{\beta k+\beta-1}s^{-\frac{\alpha}{\beta}(\beta k+\beta-1)}}{\Gamma\left(1+\frac{\alpha}{\beta}-\alpha-\alpha k\right)\Gamma(\beta+\beta k)}\right]\\
&= C_2 z^{\beta-1}s^{-\alpha+\frac{\alpha}{\beta}-1}\sum_{k=0}^{\infty}\frac{z^{\beta k}s^{-\alpha k}}{\Gamma\left(\frac{\alpha}{\beta}-\alpha-\alpha k\right)\Gamma(\beta+\beta k)}\\
&= C_2 z^{\beta-1}s^{-\alpha+\frac{\alpha}{\beta}-1}H^{1,1}_{2,2}\left[-z^{\beta}s^{-\alpha}\left|\begin{matrix}(0,1)(\frac{\alpha}{\beta}-\alpha,\alpha)\\(0,1)(1-\beta,\beta)\end{matrix}\right.\right]\\
&= \frac{C_2 z^{\beta-1}s^{-\alpha+\frac{\alpha}{\beta}-1}}{2\pi\mathrm{i}}\int_L\frac{\Gamma(-\varsigma)\Gamma(1+\varsigma)}{\Gamma(\beta+\beta\varsigma)\Gamma\left(\frac{\alpha}{\beta}-\alpha-\alpha\varsigma\right)}(-z^{\beta}s^{-\alpha})^{\varsigma}\,\mathrm{d}\varsigma,
\end{aligned}\tag{36}$$

where L is a contour separating the poles of $\Gamma(-\varsigma)$ and $\Gamma(1+\varsigma)$, i.e. $\{0,1,2,\ldots\}$ and $\{-1,-2,\ldots\}$. Here we choose L as a line from $b-\mathrm{i}\infty$ to $b+\mathrm{i}\infty$, provided that the condition $-1<b<\frac{1}{\beta}-1$ holds. Then we have

$$\begin{aligned}
{}_*P^{0,1-\alpha}_{\beta/\alpha}f(z) &= \frac{C_2 z^{\beta-1}}{\Gamma(1-\alpha)}\int_0^1(1-s)^{-\alpha}s^{\frac{\alpha}{\beta}-\alpha-1}\\
&\quad\cdot\frac{1}{2\pi\mathrm{i}}\int_L\frac{\Gamma(-\varsigma)\Gamma(1+\varsigma)}{\Gamma(\beta+\beta\varsigma)\Gamma\left(\frac{\alpha}{\beta}-\alpha-\alpha\varsigma\right)}(-z^{\beta}s^{-\alpha})^{\varsigma}\,\mathrm{d}\varsigma\,\mathrm{d}s.
\end{aligned}\tag{37}$$

Noting that $\mathrm{Re}(-\alpha\varsigma+\frac{\alpha}{\beta}-\alpha)>0$ and the definition of the *beta* function we can change the order of integration. As a result, (37) becomes

$$\begin{aligned}
&\frac{C_2 z^{\beta-1}}{2\pi\mathrm{i}}\int_L\frac{\Gamma(-\varsigma)\Gamma(1+\varsigma)}{\Gamma(\beta+\beta\varsigma)\Gamma\left(\frac{\alpha}{\beta}-\alpha-\alpha\varsigma\right)}(-z^{\beta})^{\varsigma}\cdot\frac{1}{\Gamma(1-\alpha)}\int_0^1(1-s)^{-\alpha}s^{-\alpha\varsigma+\frac{\alpha}{\beta}-\alpha-1}\,\mathrm{d}s\,\mathrm{d}\varsigma\\
&\quad= \frac{C_2 z^{\beta-1}}{2\pi\mathrm{i}}\int_L\frac{\Gamma(-\varsigma)\Gamma(1+\varsigma)}{\Gamma(\beta+\beta\varsigma)\Gamma(1-2\alpha+\frac{\alpha}{\beta}-\alpha\varsigma)}(-z^{\beta})^{\varsigma}\mathrm{d}\varsigma\\
&\quad= C_2 z^{\beta-1}H^{1,1}_{2,2}\left[-z^{\beta}\left|\begin{matrix}(0,1)(1-2\alpha+\frac{\alpha}{\beta},\alpha)\\(0,1)(1-\beta,\beta)\end{matrix}\right.\right]\\
&\quad= C_2 z^{\beta-1}W_{(-\alpha,1+\frac{\alpha}{\beta}-2\alpha)(\beta,\beta)}(z^{\beta}).
\end{aligned}\tag{38}$$

Comparing (38) with (35), we can get the result. □

Remark 3. In the process of obtaining theorem 5, the property of the Riemann–Liouville derivative ([2][chapter 2, property 2.1, p 71])

$${}^{RL}_{0}D^{\nu}_{z}z^{\nu-j}=0,\qquad j=1,2,\ldots,[\nu]+1,\tag{39}$$

is used.

Combining (23) and (34) with the boundary conditions (22), we can get the equations satisfied by the constants C_1, C_2 and p.

Theorem 6. *As for the case 1, we have*

$$\eta p^2\frac{\Gamma\left(1+\frac{\alpha}{\beta}\right)}{\Gamma\left(1+\frac{\alpha}{\beta}-\alpha\right)}W_{(-\alpha,1-\frac{\alpha}{\beta})(\beta,2)}(p^{\beta}) = p^{2-\beta}W_{(-\alpha,1-\frac{\alpha}{\beta})(\beta,3-\beta)}(p^{\beta})\tag{40}$$

and

$$C_1=\frac{1}{pW_{(-\alpha,1-\frac{\alpha}{\beta})(\beta,2)}(p^{\beta})}.\tag{41}$$

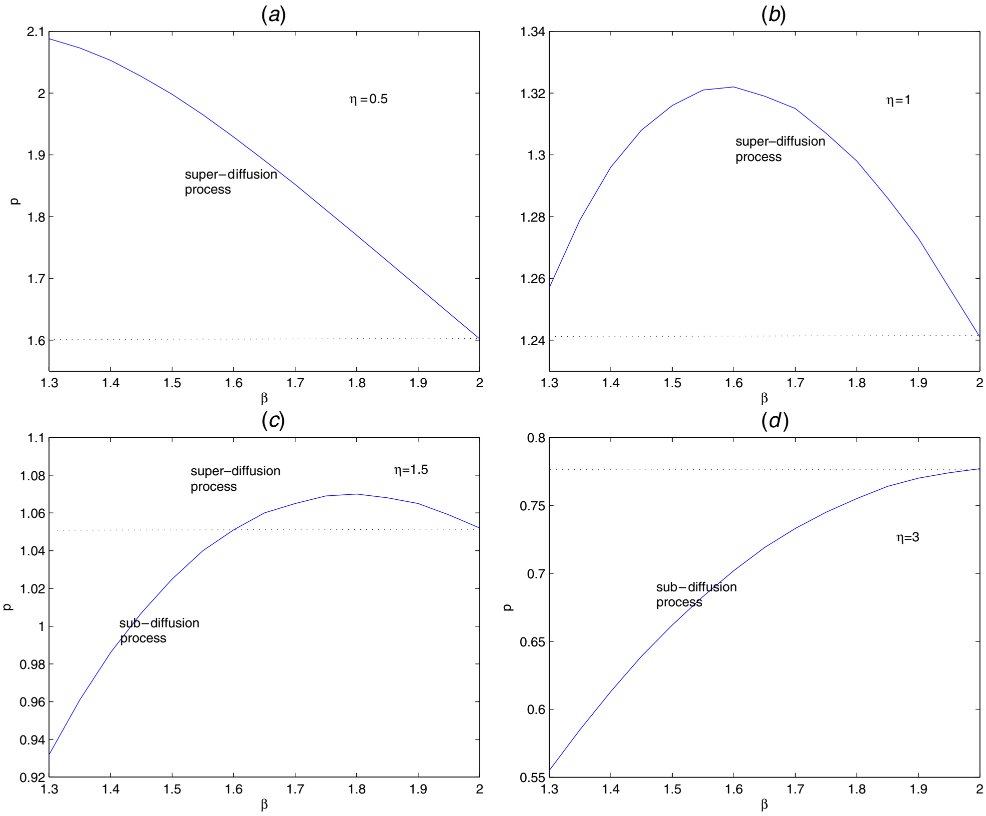

Figure 1. p versus β when the space-fractional derivative is the Caputo one. The bigger p, the faster the diffusion process.

Using the computer, the value of C_1 and p can be obtained. Correspondingly, as for the case 2, we have

$$\eta p^{\beta}\frac{\Gamma\left(1+\frac{\alpha}{\beta}\right)}{\Gamma\left(1+\frac{\alpha}{\beta}-\alpha\right)}W_{(-\alpha,1+\frac{\alpha}{\beta}-\alpha)(\beta,\beta)}(p^{\beta})=W_{(-\alpha,1+\frac{\alpha}{\beta}-\alpha)(\beta,1)}(p^{\beta}) \tag{42}$$

and

$$C_2=\frac{1}{p^{\beta-1}W_{(-\alpha,1+\frac{\alpha}{\beta}-\alpha)(\beta,\beta)}(p^{\beta})}. \tag{43}$$

4. Discussion of the solutions and conclusions

In this paper, we considered a model of a solute release from a polymer matrix in which the initial loading is higher than the solubility [12–14] and the solutions in terms of the generalized Wright function is obtained.

In the case of $\alpha=1$, $\beta=2$ the governing equation degenerate to the ordinary diffusion equation, and the model reduced to the one by Paul and McSpadden [12]. Using the formula of Gamma function, $\Gamma\left(\frac{1}{2}-n\right)=\frac{(-1)^n 2^n\sqrt{\pi}}{(2n-1)!!}$, we have $zW_{(-\alpha,1-\frac{\alpha}{\beta})(\beta,2)}(z^{\beta})=\mathrm{erf}\left(\frac{z}{2}\right)$, which is coincident with the result in [12].

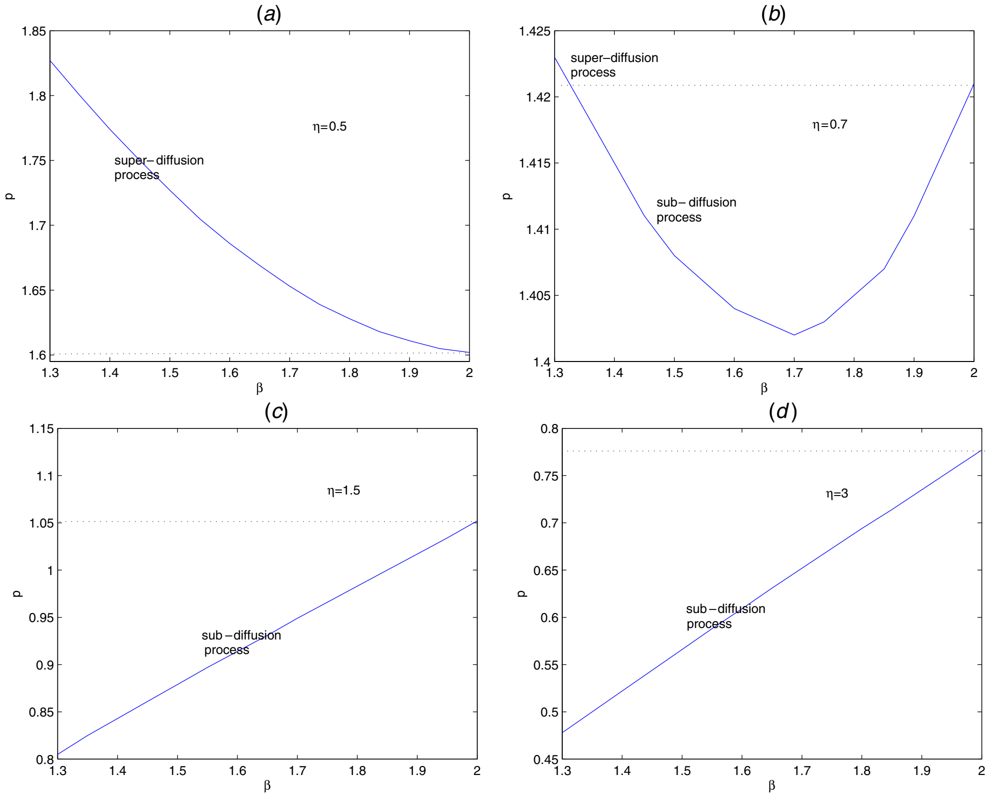

Figure 2. p versus β when the space-fractional derivative is the Riemann–Liouville one. The bigger p, the faster the diffusion process.

Table 1. Values of p in different cases.

	$\eta = 1.5$		$\eta = 3.5$		$\eta = 7$		$\eta = 10$	
(α, β)	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
(0.75, 1.75)	1.0210	0.9350	0.6430	0.5840	0.4370	0.3950	0.3570	0.3230
(0.5, 1.75)	1.0110	0.9300	0.6130	0.5580	0.4100	0.3710	0.3340	0.3020
(1, 2)	1.0520	1.0520	0.7240	0.7240	0.5230	0.5230	0.4410	0.4410
(1, 1.75)	1.0690	0.9660	0.6860	0.6180	0.4700	0.4140	0.3860	0.3480
(1, 1.5)	1.0250	0.8790	0.6000	0.5130	0.3830	0.3270	0.3030	0.2590

p is an important parameter for describing the character of the model for its value shows the velocity of the change of the moving boundary which implies the velocity of the diffusive process. The values of p in different cases are listed in table 1. Making a comparison between the values of p corresponding to two types of fractional operators, we can see that the diffusion process described by the Caputo derivative is much faster than the one by the Riemann–Liouville derivative.

However, not all the space-fractional derivatives can describe super-diffusion (or faster than ordinary diffusion) process. In order to show the influence of the order of the space-fractional derivative, we set $\alpha = 1$. The values of p versus different β in some cases are shown in figures 1 and 2. From the two figures, we can see that, for small η, e.g. $\eta = 0.5$ (figures 1(a)

and 2(a)), p is bigger in the case $\beta < 2$ than the case $\beta = 2$. This means that the Caputo and Riemann–Liouville derivatives both describe the super-diffusion process. In this case, the bigger η, the smaller p, i.e. the slower the diffusion process. This property is coincident with the one described by the Riesz derivative. For larger η, for example, $\eta = 3$ (figures 1(d) and 2(d)), both the two derivatives cannot describe the super-diffusion process, while at some cases (figures 1(c) and 2(b)), the two operators can conditionally describe the super-diffusion process.

The spatial fractional diffusion equations are mainly used to describe the super-diffusion processes. However, with a brief analysis of the solutions, we demonstrated that the Caputo and Riemann–Liouville fractional derivatives cannot always do the business. Therefore, when using these two derivatives in describing the anomalous diffusions, the initial loading must be considered.

Acknowledgments

The authors wish to express their thanks to the anonymous referees for their fruitful advices and comments. The author XC Li wishes to express his thanks to Professor CY Wang (Professor of Michigan State University) for his useful discussion. This work was supported by the Natural Science Foundation of the Shandong Province of China (grant no Y2007A06).

References

[1] Podlubny I 1999 *Fractional Differential Equations* (New York: Academic)
[2] Kilbas A A, Srivastava H M and Trujillo J J 2006 *Theory and Applications of Fractional Differential Equations* (Amsterdam: Elsevier)
[3] Adolfsson K, Enelund M and Olsson P 2005 On the fractional order model of viscoelasticity *Mech. Time-Dependent Mater.* **9** 15–34
[4] El-Shahed M and Salem A 2006 Decay of vortex velocity and diffusion of temperature for fractional viscoelastic fluid through a porous medium *Int. Commun. Heat Mass Transfer* **33** 240–8
[5] Davis G B, Kohandel M, Sivaloganathan S and Tenti G 2006 The constitutive properties of the brain paraenchyma: part 2. Fractional derivative approach *Med. Eng. Phys.* **28** 455–9
[6] Metzler R and Klafter J 2000 The random walk's guide to anomalous diffusion: a fractional dynamics approach *Phys. Rep.* **339** 1–77
[7] Metzler R and Klafter J 2004 The restaurant at the end of the random walk: recent development in the description of anomalous transport by fractional dynamics *J. Phys. A: Math. Gen* **37** 161–208
[8] Mainardi F, Luchko Y and Pagnini G 2001 The fundamental solution of the space-time fractional diffusion equation *Fract. Calc. Appl. Anal.* **4** 153–92
[9] Xu M Y and Tan W C 2006 Intermediate processes and critical phenomena: theory, method and progress of fractional operators and their applications to modern mechanics *Sci. Series Ser. G: Phys. Mech. Astron.* **49** 257–72
[10] Metzler R and Klafter J 2000 Boundary value problems for fractional diffusion *Physica* A **278** 107–25
[11] Schneider W R and Wyss W 1989 Fractional diffusion and wave equation *J. Math. Phys.* **30** 134–44
[12] Paul D R and McSpadden S K 1976 Diffusional release of a solute from a polymer matrix *J. Membr. Sci.* **1** 33–48
[13] Liu J Y and Xu M Y 2004 An exact solution to the moving boundary problem with fractional anomalous diffusion in drug release devices *Z. Angew. Math. Mech.* **84** 22–8
[14] Li X C, Xu M Y and Wang S W 2007 Analytical solutions to the moving boundary problems with time-space-fractional derivatives in drug release devices *J. Phys. A: Math. Theor.* **40** 12131–41
[15] Krepysheva N, Pietro L D and Neel M C 2006 Space-fractional advection-diffusion and reflective boundary condition *Phys. Rev.* E **73** 021104
[16] Benson D A *et al* 2004 Radial fractional-order dispersion through fractured rock *Water Resour. Res.* **40** W12416
[17] Liu F, Anh V and Turner I 2004 Numerical solution of fractional Fokker-Planck equation *J. Comput. Appl. Math.* **166** 209–19
[18] del-Castillo-Negrete D 2006 Fractional diffusion models of nonlocal transport *Phys. Plasmas* **13** 082308

[19] Zhang X *et al* 2007 The impact of boundary on the fractional advection-dispersion equation for solute transport in soil: defining the fractional dispersive flux with Caputo derivatives *Adv. Water Resour.* **30** 1205–17

[20] Crank J 1987 *Free and Moving Boundary Problems* (Oxford: Clarendon)

[21] Luchko Y and Gorenflo R 1998 Scale-invariant solutions of a partial differential equation of fractional order *Fract. Calc. Appl. Anal.* **1** 63–78

[22] Buckwar E and Luchko Y 1998 Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations *J. Math. Anal. Appl.* **227** 81–97

[23] Gorenflo R, Luchko Y and Mainardi F 2000 Wright function as scale-invariant solutions of the diffusion-wave equation *J. Comp. Appl. Math.* **118** 175–91

[24] Mathai A M and Saxena R K 1978 *The H function with applications in statistics and other disciplines* (New Delhi: Wiley Easterm Ltd)

[25] Wyss W 1986 The fractional diffusion equation *J. Math. Phys.* **27** 2782–5